
On the Generality of the Greedy Algorithm for Solving Matroid Base Problems

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Abstract It is well known that the greedy algorithm solves matroid base problems for all linear cost functions and is, in fact, correct if and only if the underlying combinatorial structure of the problem is a matroid. Moreover, the algorithm can be applied to problems with sum, bottleneck, algebraic sum or k -sum objective functions.

In this paper, we address matroid base problems with a more general – “universal” – objective function which contains the previous ones as special cases. This universal objective function is of the sum type and associates multiplicative weights with the ordered cost coefficients of the elements of matroid bases such that, by choosing appropriate weights, many different – classical and new – objectives can be modeled. We show that the greedy algorithm is applicable to a larger class of objective functions than commonly known and, as such, it solves universal matroid base problems with non-negative or non-positive weight coefficients. Based on problems with mixed weights and a single $(-, +)$ -sign change in the universal weight vector, we give a characterization of uniform matroids. In case of multiple sign changes, we use partition matroids. For non-uniform matroids, single sign change problems can be reduced to problems in minors obtained by deletion and contraction. Finally, we discuss how special instances of universal bipartite matching and shortest

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path problems can be tackled by applying greedy algorithms to associated transversal matroids.

Keywords Combinatorial optimization · Matroids · Universal objective function · Greedy algorithm · Uniform matroids · Minors · Transversal matroids · Bipartite matchings · Shortest paths

Mathematics Subject Classification (2000) 05B35 · 05C38 · 05C70 · 05C85 · 05D15 · 90C27

1 Introduction

Quoting Welsh [26, p. v], matroids play a “unifying and central role ... in combinatorial theory” as various problems in combinatorial optimization can be modeled as matroid base problems and, therefore, be solved efficiently by greedy algorithms. For a matroid $M = (E, \mathcal{B})$ given by its ground set E and its collection of bases \mathcal{B} , the classical problem is the *minimum matroid base problem (MMBP)*

$$\min_{B \in \mathcal{B}} \sum_{e \in B} c(e) \quad (1)$$

where costs $c(e) \in \mathbb{R}$ are assigned to the elements $e \in E$. It determines a matroid base $B^* \in \mathcal{B}$ such that the sum of its cost coefficients is minimal. A base $B^* \in \mathcal{B}$, which is optimal to (1), is called a *minimum-cost base*.

The goal of this paper is to show that the greedy algorithm – which starts with the empty set and iteratively adds an element of smallest possible cost while preserving independence of the set – is not only correct for matroid base problems with sum objective function, but also for many other objectives. Among them are, for instance, bottleneck, algebraic sum, k -sum, k -max, cent-dian and trimmed-mean objective functions.

To this end, the remainder of the paper is organized as follows:

In Section 2, we introduce a unified framework to formulate matroid base problems with different types of objective functions. Based on ordered weighted averaging operators and ordered median functions, these universal matroid base problems are sum objectives where multiplicative weights are associated with the ordered cost coefficients of the elements of matroid bases. The solvability by greedy methods will heavily depend on the chosen weight coefficients.

In Section 3, we review the (standard) greedy algorithm. Its power is illustrated in Section 4 where universal matroid base problems with either non-negative or non-positive weight coefficients are solved. As opposed to this, we give an example to demonstrate that the “pure” greedy strategy fails if there are strictly positive and strictly negative weights. Consequently, the subsequent sections focus on problems with mixed weights.

We distinguish between single (Section 5) and multiple (Section 6) sign changes in the universal weight vectors as well as between uniform and non-uniform matroids. For uniform matroids, we prove the result that a matroid base composed by a minimum-cost and maximum-cost base is optimal for

problems with one $(-, +)$ -sign change. We also show that this is only true for uniform matroids, thus providing a new characterization of this matroid class. These results can be carried over to problems with multiple sign changes, however, a solution in appropriately defined partition matroids becomes necessary. For non-uniform matroids, we use deletion and contraction operations if the universal weight vector changes its sign only once.

In Section 7, we consider two applications of transversal matroids, the universal bipartite matching problem and the universal shortest path problem. Our results are summarized in Section 8.

In the appendix, notations as well as basic definitions and results from matroid theory are summarized.

2 Universal Matroid Bases

Given a matroid $M = (E, \mathcal{B})$, the focus is on matroid base problems with universal objective function generalizing the well-known sum objective function and including several more as special cases.

The definition is based on two parts, the sorting of the cost coefficients of the elements of bases $B \in \mathcal{B}$ and their multiplication with universal weights $\lambda_1, \dots, \lambda_r$. The rank of matroid $M = (E, \mathcal{B})$, which is equal to the cardinality of its bases, is denoted by r .

Definition 1 Given costs $c(e) \in \mathbb{R}$ for all elements $e \in E$ and a base $B \in \mathcal{B}$, the *sorted cost vector* (with respect to $c(e)$, $e \in E$, and B) is

$$c_{\geq}(B) := (c_{(1)}(B), \dots, c_{(r)}(B))$$

where $c_{(i)}(B)$, $i = 1, \dots, r$, is the i th largest cost coefficient of base B .

Combining the sorted costs with a given set of weights, we get a universal objective function.

Definition 2 Given a matroid $M = (E, \mathcal{B})$ of rank r with

- costs $c(e) \in \mathbb{R}$ for all $e \in E$ and
- weights $\lambda_i \in \mathbb{R}$ for all $i = 1, \dots, r$,

the *universal minimum matroid base problem* (*Univ-MMBP*) is

$$\min_{B \in \mathcal{B}} f_{\lambda}(B) := \sum_{i=1}^r \lambda_i \cdot c_{(i)}(B). \quad (2)$$

An optimal base $B^* \in \mathcal{B}$ is called a *universal minimum-cost base*.

According to Definition 2, the universal objective function $f_{\lambda}(\cdot)$ is the scalar product of the universal weight vector $\lambda = (\lambda_1, \dots, \lambda_r)$ and the sorted cost vector $c_{\geq}(B) = (c_{(1)}(B), \dots, c_{(r)}(B))$. As such, it is compatible with ordered weighted averaging (OWA) operators and ordered median functions

which have been introduced by Yager [27] and Nickel and Puerto [17] in order to aggregate criteria functions in multicriteria decision-making and to model flexible objectives in location theory, respectively.

To call problem (2) “universal” truly makes sense, since it combines the numerous special cases of matroid base problems with a variety of objectives induced by specific choices of weight coefficients $\lambda_i \in \mathbb{R}$, $i = 1, \dots, r$. This includes the well-known cases of *sum objective*

$$\min_{B \in \mathcal{B}} \sum_{e \in B} c(e)$$

and *bottleneck objective*

$$\min_{B \in \mathcal{B}} \max_{e \in B} c(e)$$

which can be seen by choosing $\lambda_i = 1$ for all $i = 1, \dots, r$ or $\lambda_1 = 1$, $\lambda_i = 0$ otherwise, respectively.

The *balanced objective function* studied by Duin and Volgenant [6] or Martello et al. [15] in which we minimize the difference between the largest and smallest cost coefficient, i.e.

$$\min_{B \in \mathcal{B}} \left(\max_{e \in B} c(e) - \min_{e \in B} c(e) \right)$$

is obtained for $\lambda_1 = 1$, $\lambda_r = -1$ and $\lambda_i = 0$ otherwise. The (k_1, k_2) -*balanced objective function* which is a new variant and minimizes the difference between the k_1 th largest and the k_2 th smallest cost coefficient is modeled with $\lambda_{k_1} = 1$, $\lambda_{r-k_2+1} = -1$ and $\lambda_i = 0$ otherwise (see Turner [22, 23]).

We can formulate matroid base problems with *algebraic sum*, i.e. combined min-max min-sum, *objective function*

$$\min_{B \in \mathcal{B}} \left(\max_{e \in B} c(e) + \sum_{e \in B} c(e) \right)$$

or *minimum deviation objective function*

$$\min_{B \in \mathcal{B}} \sum_{e_i \in B} \left(\max_{e \in B} c(e) - c(e_i) \right)$$

by setting $\lambda_1 = 2$, $\lambda_i = 1$ otherwise or $\lambda_1 = r - 1$, $\lambda_i = -1$ otherwise. For general combinatorial optimization problems, algebraic sum objectives are studied in Minoux [16] and Punnen [19]. For minimum deviation problems, we refer to Gupta and Punnen [11] or Duin and Volgenant [6].

The sum of the k largest cost coefficients or the k th largest cost coefficient itself is minimized in k -*sum objectives* (see Gupta and Punnen [12] or Punnen and Aneja [20]) or k -*max objectives* (see Gorski and Ruzika [10]) where $\lambda_1 = \dots = \lambda_k = 1$ or $\lambda_k = 1$, respectively, while $\lambda_i = 0$ otherwise. Similarly, we choose $\lambda_{k_1} = \lambda_{k_2} = 1$ and $\lambda_i = 0$ otherwise to model (k_1, k_2) -*max objective functions* where the sum of the k_1 th largest and k_2 th largest cost coefficient is considered (compare Turner [22, 23]).

The *cent-dian* and *anti-cent-dian objective functions*, which are known from location theory, are special cases obtained for $\lambda = (1, \alpha, \dots, \alpha)$ and $\lambda = (-1, -\alpha, \dots, -\alpha)$ where $\alpha \in \mathbb{R}_0^+$.

Finally, for the $(k_1 + k_2)$ -*trimmed-mean objective function*, in which the k_1 largest and k_2 smallest cost coefficients of the bases $B \in \mathcal{B}$ are ignored and the remaining cost coefficients are added, we set $\lambda_{k_1+1} = \dots = \lambda_{r-k_2} = 1$ and $\lambda_i = 0$ otherwise. In contrast, the $(k_1 + k_2)$ -*anti-trimmed-mean objective function*, in which all but the k_1 largest and k_2 smallest cost coefficients of the bases $B \in \mathcal{B}$ are ignored, is modeled via $\lambda_1 = \dots = \lambda_{k_1} = 1$, $\lambda_{r-k_2+1} = \dots = \lambda_r = 1$ and $\lambda_i = 0$ otherwise.

An overview on the objectives modeled by Univ-MMBP is given in Table 1 (compare Nickel and Puerto [17]). This list is not exhaustive and focuses on weight coefficients in $\{0, \pm 1\}$ although many other objective functions with arbitrary real-valued weights (e.g., non-increasing or non-decreasing weights, weights in blocks, ...) are possible. The universal matroid base problem is a special case of the *universal combinatorial optimization problem (Univ-COP)* introduced in Turner [23].

Apart from OWA operators, ordered median functions and universal objectives, other approaches to develop unifying theories in mathematical optimization have been proposed including the concepts of algebraic optimization (see Burkard and Hamacher [5], Hamacher [13] or Zimmermann [28]) or of discrete optimization with ordering (see Fernández et al. [7, 8]).

The following observation will be helpful in the next sections.

Lemma 1 *Given a base $B := \{b_1, \dots, b_r\} \in \mathcal{B}$ with $c(b_1) \leq \dots \leq c(b_r)$ or $c(b_1) \geq \dots \geq c(b_r)$, Univ-MMBP can be reformulated as*

$$\min_{B \in \mathcal{B}} f_\lambda(B) := \sum_{i=1}^r \lambda_i \cdot c(b_{r-i+1}) \quad (3a)$$

or

$$\min_{B \in \mathcal{B}} f_\lambda(B) := \sum_{i=1}^r \lambda_i \cdot c(b_i), \quad (3b)$$

respectively.

3 The Greedy Algorithm

Before we show the generality of the greedy algorithm in solving universal matroid base problems, we start with a repetition of its basic functionality.

Consider a matroid $M = (E, \mathcal{I})$ with ground set E and independence system \mathcal{I} . The *greedy algorithm* which generalizes Kruskal's algorithm for the minimum spanning tree problem (MSTP) starts with the independent set $I_0 := \emptyset$ and chooses iteratively an element $e_{i_{k+1}} \notin I_k$ of smallest possible cost if and only if no circuit is generated by its addition to subset $I_k \in \mathcal{I}$. For all cost functions $c: E \rightarrow \mathbb{R}$, it computes a minimum-cost base $B^* \in \mathcal{B}$ to (1).

Table 1 Objective functions modeled by Univ-MMBP

Name	Type	Universal weight vector
Sum	$\sum_{e \in B} c(e)$	$(1, \dots, 1)$
Bottleneck	$\max_{e \in B} c(e)$	$(1, 0, \dots, 0)$
Balanced	$\max_{e \in B} c(e) - \min_{e \in B} c(e)$	$(1, 0, \dots, 0, -1)$
(k_1, k_2) -balanced	$c_{(k_1)}(B) - c_{(r-k_2+1)}(B)$	$(\underbrace{0, \dots, 0}_{k_1-1}, 1, 0, \dots, 0, -1, \underbrace{0, \dots, 0}_{k_2-1})$
Algebraic sum	$\max_{e \in B} c(e) + \sum_{e \in B} c(e)$	$(2, 1, \dots, 1)$
Minimum deviation	$\sum_{e_i \in B} (\max_{e \in B} c(e) - c(e_i))$	$(r-1, -1, \dots, -1)$
k -sum	$\sum_{i=1}^k c_{(i)}(B)$	$(\underbrace{1, \dots, 1}_k, 0, \dots, 0)$
k -max	$c_{(k)}(B)$	$(\underbrace{0, \dots, 0}_{k-1}, 1, 0, \dots, 0)$
(k_1, k_2) -max	$c_{(k_1)}(B) + c_{(k_2)}(B)$	$(\underbrace{0, \dots, 0}_{k_1-1}, 1, 0, \dots, 0, \underbrace{1, 0, \dots, 0}_{k_2-1})$
Cent-dian	$(1 - \alpha) \cdot \max_{e \in B} c(e) + \alpha \cdot \sum_{e \in B} c(e)$	$(1, \alpha, \dots, \alpha)$
Anti-cent-dian	$(\alpha - 1) \cdot \max_{e \in B} c(e) - \alpha \cdot \sum_{e \in B} c(e)$	$(-1, -\alpha, \dots, -\alpha)$
$(k_1 + k_2)$ -trimmed-mean	$\sum_{i=k_1+1}^{r-k_2} c_{(i)}(B)$	$(\underbrace{0, \dots, 0}_{k_1}, 1, \dots, 1, \underbrace{0, \dots, 0}_{k_2})$
$(k_1 + k_2)$ -anti-trimmed-mean	$\sum_{i=1}^{k_1} c_{(i)}(B) + \sum_{i=r-k_2+1}^r c_{(i)}(B)$	$(\underbrace{1, \dots, 1}_{k_1}, 0, \dots, 0, \underbrace{1, \dots, 1}_{k_2})$

The solution found by the greedy algorithm is the unique optimal solution to MMBP if the costs $c(e) \in \mathbb{R}$ are different for all elements $e \in E$. Any minimum-cost base $B^* \in \mathcal{B}$ can be found by this strategy.

In the same way, these observations hold for maximum-cost bases $B^* \in \mathcal{B}$ which are optimal for the maximization problem

$$\max_{B \in \mathcal{B}} \sum_{e \in B} c(e),$$

in which the sum of the cost coefficients is maximized instead of minimized. A maximum-cost base $B^* \in \mathcal{B}$ can be determined similarly if we choose an element $e_{i_{k+1}} \notin I_k$ of largest possible cost in the greedy algorithm.

Note that the formulation of the greedy algorithm (see Oxley [18]) includes the minimization of the sum objective as standard and its maximization as alternative. In the following, we refer to these versions as min or max version of Algorithm 1, respectively.

Algorithm 1 Greedy algorithm

Input: Matroid $M = (E, \mathcal{I})$ with costs $c(e) \in \mathbb{R}$.

1: Initialize $I_0 := \emptyset$ and $k := 0$.

2: **while** $k < r$ **do**

3: Choose an element $e_{i_{k+1}} \notin I_k$ of smallest possible cost such that $I_k + e_{i_{k+1}} \in \mathcal{I}$.

// **Alternative:** largest possible cost

4: Set $I_{k+1} := I_k + e_{i_{k+1}}$ and $k := k + 1$.

5: **end while**

6: Let $B^* := I_r$.

Output: Minimum-cost base $B^* \in \mathcal{B}$.

// **Alternative:** Maximum-cost base

Next, we establish the component-wise optimality of the greedy algorithm which can, for instance, be found in Oxley [18].

Lemma 2 *Let $B^* := \{b_1^*, \dots, b_r^*\}$ be a base obtained from the min version of Algorithm 1 by iteratively choosing the elements b_1^*, \dots, b_r^* and let $B := \{b_1, \dots, b_r\}$ be any other base with $c(b_1) \leq \dots \leq c(b_r)$. Then,*

$$c(b_i^*) \leq c(b_i)$$

holds for all $i = 1, \dots, r$.

Applying Lemma 2, we immediately see that the greedy algorithm solves matroid base problems with sum and bottleneck objective function. The same has been observed by Gupta and Punnen [12] for k -sum objectives and by Punnen [19] for algebraic sum, i.e. combined min-max min-sum, objectives. Since the above optimality does not depend on the absolute values, but only on the relative ordering of the costs, it seems plausible that the greedy algorithm is well-suited to tackle universal matroid base problems in which the cost coefficients are sorted before they are weighted. That this conjecture is correct for large classes of universal weights will be shown in the next sections.

4 Universal Weights without Sign Changes

As a consequence of Lemma 2, the greedy algorithm is valid for Univ-MMBP if the weight coefficients satisfy sign constraints, thus generalizing a result by Fernández et al. [7] for ordered median minimum spanning tree problems.

Theorem 1 *If all weight coefficients are*

- *non-negative, i.e. $\lambda_1, \dots, \lambda_r \geq 0$, or*
- *non-positive, i.e. $\lambda_1, \dots, \lambda_r \leq 0$,*

Univ-MMBP is solvable by Algorithm 1.

Proof We consider a base $B^* := \{b_1^*, \dots, b_r^*\}$ found by the min version of Algorithm 1 and any other base $B := \{b_1, \dots, b_r\}$ with $c(b_1) \leq \dots \leq c(b_r)$. If all weight coefficients are non-negative, we conclude from Lemma 2 that

$$\lambda_i \cdot c(b_{r-i+1}^*) \leq \lambda_i \cdot c(b_{r-i+1})$$

for all $i = 1, \dots, r$ such that

$$f_\lambda(B^*) = \sum_{i=1}^r \lambda_i \cdot c(b_{r-i+1}^*) \leq \sum_{i=1}^r \lambda_i \cdot c(b_{r-i+1}) = f_\lambda(B).$$

Analogously, for a base $B^* := \{b_1^*, \dots, b_r^*\}$ found by the max version of Algorithm 1 and any other base $B := \{b_1, \dots, b_r\}$ with $c(b_1) \geq \dots \geq c(b_r)$, it holds that $c(b_i^*) \geq c(b_i)$ for all $i = 1, \dots, r$. We conclude that

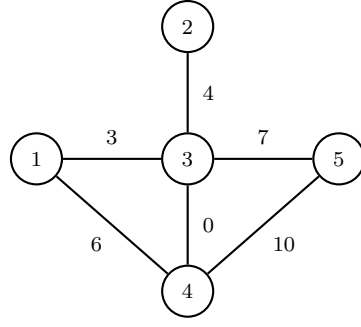
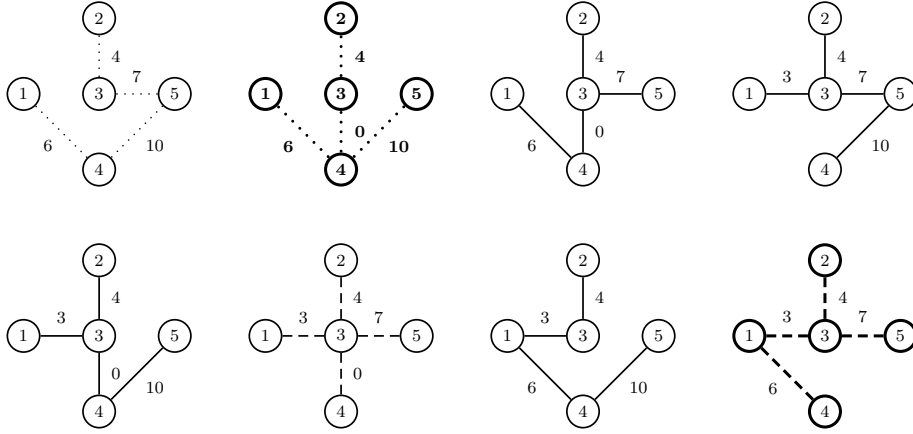
$$f_\lambda(B^*) = \sum_{i=1}^r \lambda_i \cdot c(b_i^*) \leq \sum_{i=1}^r \lambda_i \cdot c(b_i) = f_\lambda(B)$$

when multiplying with non-positive weight coefficients. \square

Note that, except for balanced, (k_1, k_2) -balanced and minimum deviation objectives, Theorem 1 applies to all objective functions listed in Table 1. Here, sum, bottleneck, algebraic sum, k -sum and k -max, cent-dian, $(k_1 + k_2)$ -trimmed-mean and $(k_1 + k_2)$ -anti-trimmed-mean objectives have non-negative weights while anti-cent-dian objectives have non-positive weights.

The next example shows that changes of signs in the universal weights destroy the validity of the greedy algorithm, even if the weight coefficients are monotone.

Example 1 We consider the cycle matroid associated with the undirected graph $G = (V, E)$ and its spanning trees $T \in \mathcal{T}$ (see Figures 1 and 2).

Fig. 1 Graph $G = (V, E)$ Fig. 2 Spanning trees $T \in \mathcal{T}$

In graph $G = (V, E)$, the spanning trees found by the min and max version of Algorithm 1 are $T' = \{[1, 3], [2, 3], [3, 4], [3, 5]\}$ (non-bold, dashed) and $T'' = \{[1, 4], [2, 3], [3, 5], [4, 5]\}$ (non-bold, dotted), respectively. For $\lambda' = (2, 1, -1, -2)$, the universal costs are $f_{\lambda'}(T') = 15$ and $f_{\lambda'}(T'') = 13$; for $\lambda'' = (-2, -1, 1, 2)$, they are $f_{\lambda''}(T') = -15$ and $f_{\lambda''}(T'') = -13$. These spanning trees are not optimal since $f_{\lambda'}(T^*) = 10$ and $f_{\lambda''}(T^{**}) = -22$, where $T^* = \{[1, 3], [1, 4], [2, 3], [3, 5]\}$ and $T^{**} = \{[1, 4], [2, 3], [3, 4], [4, 5]\}$. In Figure 2, the universal minimum spanning trees T^* (dashed) and T^{**} (dotted) have bold edges.

5 Universal Weights with a Single Sign Change

As shown in Example 1, the greedy algorithm (minimization or maximization) in its pure form is not sufficient to solve universal matroid base problems if

the weight coefficients switch signs. In this section, we study universal weight vectors with only one sign change.

Definition 3 A universal weight vector has *minus-plus form* if

$$\lambda = (-\alpha_1, \dots, -\alpha_k, \beta_{k+1}, \dots, \beta_r)$$

where $\alpha_1, \dots, \alpha_k \in \mathbb{R}_0^+$ and $\beta_{k+1}, \dots, \beta_r \in \mathbb{R}_0^+$ with at least one weight coefficient $\alpha_i > 0$ and one $\beta_j > 0$. The resulting universal matroid base problem is denoted as *minus-plus Univ-MMBP*.

Minus-plus Univ-MMBP is more general than the problem presented in Example 1 since we do not assume monotonicity of the weight coefficients.

5.1 Uniform Matroids

In this subsection, we show that minus-plus Univ-MMBP is solvable by a combination of the min and max version of the greedy algorithm if and only if we assume that the underlying matroid is uniform. The idea is to associate the $r - k$ smallest cost coefficients of a minimum-cost base with the non-negative weights and the k largest cost coefficients of a maximum-cost base with the non-positive weights.

Given a uniform matroid $M_{r,m} = (E(M_{r,m}), \mathcal{I}(M_{r,m}))$ of rank r , the bases $B \in \mathcal{B}(M_{r,m})$ are all subsets with exactly r elements. The independent sets are all subsets $I \subseteq E(M_{r,m})$ with at most r elements (see Appendix).

Theorem 2 *Let*

- a uniform matroid $M_{r,m} = (E(M_{r,m}), \mathcal{I}(M_{r,m}))$,
- a weight vector $\lambda \in \mathbb{R}^r$ of *minus-plus form*,
- a minimum-cost base $B' := \{b'_1, \dots, b'_r\}$ with $c(b'_1) \leq \dots \leq c(b'_r)$ and a maximum-cost base $B'' := \{b''_1, \dots, b''_r\}$ with $c(b''_1) \geq \dots \geq c(b''_r)$ found by the min and max version of Algorithm 1, respectively, be given.

Then, $B^ := \{b^*_1, \dots, b^*_r\}$ with $b^*_i := b''_i$ for all $i = 1, \dots, k$ and $b^*_i := b'_{r-i+1}$ for all $i = k+1, \dots, r$ is an optimal solution of minus-plus Univ-MMBP.*

Proof By definition of uniform matroids, the combination of the k -element subset $\{b''_1, \dots, b''_k\}$ and the $(r - k)$ -element subset $\{b'_1, \dots, b'_{r-k}\}$ is a base. Applying Lemma 2 to the bases B' and B'' , taking into account that their elements are sorted in the reverse order, we have that $c(b''_k) \geq c(b'_{r-k+1})$ and, thus,

$$c(b''_1) \geq \dots \geq c(b''_k) \geq c(b'_{r-k}) \geq \dots \geq c(b'_1).$$

Since $c(b'_i) \geq c(b_i)$, $i = 1, \dots, k$, and $c(b'_{r-i+1}) \leq c(b_i)$, $i = k+1, \dots, r$, for any other base $B := \{b_1, \dots, b_r\}$ with $c(b_1) \geq \dots \geq c(b_r)$, it follows that

$$\begin{aligned} f_\lambda(B^*) &= \sum_{i=1}^k -\alpha_i \cdot c(b_i^*) + \sum_{i=k+1}^r \beta_i \cdot c(b_i^*) \\ &= \sum_{i=1}^k -\alpha_i \cdot c(b''_i) + \sum_{i=k+1}^r \beta_i \cdot c(b'_{r-i+1}) \\ &\leq \sum_{i=1}^k -\alpha_i \cdot c(b_i) + \sum_{i=k+1}^r \beta_i \cdot c(b_i) = f_\lambda(B). \end{aligned}$$

Hence, $B^* := \{b_1^*, \dots, b_r^*\}$ is a universal minimum-cost base. \square

Theorem 2 ensures the validity of Algorithm 2.

Algorithm 2 Algorithm for minus-plus Univ-MMBP in uniform matroids

Input: Uniform matroid $M_{r,m} = (E(M_{r,m}), \mathcal{I}(M_{r,m}))$ with costs $c(e) \in \mathbb{R}$.

- 1: Determine a minimum-cost base $B' \in \mathcal{B}(M_{r,m})$.
- 2: Determine a maximum-cost base $B'' \in \mathcal{B}(M_{r,m})$.
- 3: Set $b_i^* := b'_i$ for all $i = 1, \dots, k$ and $b_i^* := b'_{r-i+1}$ for all $i = k+1, \dots, r$.
- 4: Let $B^* := \{b_1^*, \dots, b_r^*\}$.

Output: Universal minimum-cost base $B^* \in \mathcal{B}(M_{r,m})$.

The uniformity of the given matroid is essential in the proof of Theorem 2 because, in non-uniform matroids, the subset $B^* := \{b_1^*, \dots, b_r^*\}$ is usually not an independent set. In particular, it is not a base. This can, for instance, be seen in Example 1 where, for weight vector $\lambda'' = (-2, -1, 1, 2)$, the union of the two smallest cost edges in tree T' and the two largest cost edges in tree T'' contains the cycle $C = \{[3, 4], [3, 5], [4, 5]\}$.

Under the mild assumption of loop-free matroids, we can show even more by establishing that Theorem 2 is correct if and only if the matroid is uniform.

Theorem 3 *Let $M = (E, \mathcal{I})$ be a matroid of rank r without loops. If, for all weight vectors of minus-plus form, an optimal solution to Univ-MMBP can be obtained by Algorithm 2, then the matroid is uniform.*

Proof Suppose that the matroid is non-uniform. There exists an r -element subset $X := \{e_1, \dots, e_r\} \subseteq E$ which is dependent. Now, let $k \in \{1, \dots, r\}$ be minimal with the property that $\{e_1, \dots, e_{k-1}\}$ is independent and $\{e_1, \dots, e_{k-1}, e_k\}$ is dependent. We have that $k > 1$ and $r > 1$ since the matroid has no loops. Using the extension property for independent sets (see Appendix), the subset $\{e_1, \dots, e_{k-1}\}$ can be extended to the independent set

$$X' := \{e_1, \dots, e_{k-1}, e'_{k+1}, \dots, e'_r\}$$

of cardinality $r - 1$ where $X' = \{e_1, \dots, e_{r-1}\}$ if $k = r - 1$. Adding element e_k , we get (another) dependent set $X^* := X' + e_k$ of cardinality r . Note that e_k is not a loop, and, therefore, both, $\{e_1, \dots, e_{k-1}, e'_{k+1}, \dots, e'_r\}$ and $\{e_k\}$, are independent sets. If we assign costs of value 1 to elements $e_1, \dots, e_{k-1}, e'_{k+1}, \dots, e'_r$ and a cost of value -1 to element e_k while the remaining cost values are set to zero, we have that $e_k \in B'$ and $e_1, \dots, e_{k-1}, e'_{k+1}, \dots, e'_r \in B''$ where B' and B'' are the minimum-cost and maximum-cost bases found by Algorithm 2. Theorem 2 would produce the dependent set $X^* \subseteq E$ as an optimal solution to Univ-MMBP with weight vector $\lambda = (-1, \dots, -1, 1)$ of minus-plus form. This is a contradiction. \square

Altogether, Theorems 2 and 3 yield a characterization of uniform matroids.

Corollary 1 *A matroid $M = (E, \mathcal{I})$ of rank r without loops is a uniform matroid if and only if, for all weight vectors of minus-plus form, we obtain an optimal solution to Univ-MMBP by Algorithm 2.*

5.2 Non-Uniform Matroids

In Example 1, we can further observe that, for weight vector $\lambda'' = (2, 1, -1, -2)$, the universal minimum spanning tree $T^* = \{[1, 3], [1, 4], [2, 3], [3, 5]\}$ is greedy optimal as long as the edges $\{[1, 4], [3, 5]\}$ are fixed as the two largest cost edges.

To adapt this idea to non-uniform matroids, we assume that the k largest cost elements of the bases are already chosen and compute the $r - k$ smallest cost elements by a greedy algorithm. In the following, we will show how this can be realized by using deletion and contraction operations. For simplicity, we assume without loss of generality that all cost coefficients $c(e)$, $e \in E$, are different.

Lemma 3 *Let*

- a matroid $M = (E, \mathcal{I})$,
- a weight vector $\lambda \in \mathbb{R}^r$ of minus-plus form,
- an independent set $B'' := \{b''_1, \dots, b''_k\}$ with $c(b''_1) > \dots > c(b''_k)$ and
- a minimum-cost independent set $B' := \{b'_1, \dots, b'_{r-k}\}$ with $c(b'_{r-k}) < c(b''_k)$ and $c(b'_1) < \dots < c(b'_{r-k})$ found by the min version of Algorithm 1 be given.

If $B^ := \{b^*_1, \dots, b^*_r\}$ with $b^*_i := b''_i$ for all $i = 1, \dots, k$ and $b^*_i := b'_{r-i+1}$ for all $i = k + 1, \dots, r$ is a base, it is an optimal solution of minus-plus Univ-MMBP among all bases $B \in \mathcal{B}$ with the elements b''_1, \dots, b''_k as k largest cost elements.*

Proof Similar to the proof of Theorem 2, because the assumptions ensure that

$$c(b^*_1) > \dots > c(b^*_k) > c(b^*_{k+1}) > \dots > c(b^*_r)$$

and $c(b^*_i) \leq c(b_i)$, $i = k + 1, \dots, r$, for any other base $B \in \mathcal{B}$ having the elements b''_1, \dots, b''_k as k largest cost elements. \square

The minimum-cost independent set $B' := \{b'_1, \dots, b'_{r-k}\}$ needed in Lemma 3 can be found (if it exists) by applying the min version of the greedy algorithm to the minor $M \setminus E''/B''$ which is obtained from matroid $M = (E, \mathcal{I})$ by contracting the independent set $B'' := \{b''_1, \dots, b''_k\}$ with $c(b''_1) > \dots > c(b''_k)$ and deleting the elements in subset

$$E'' := \{e \in E: c(e) \geq c(b''_k)\} \setminus B''.$$

By this choice, we guarantee two things: Firstly, the elements in subset B'' are the k largest elements of the bases $B^* \in \mathcal{B}$ and, secondly, the elements in subset B' do not interfere with this ordering. They are the $r - k$ smallest elements.

If the roles of subsets B' and B'' are interchanged, the max version of the greedy algorithm can be applied to the minor $M \setminus E'/B'$ where the independent set $B' := \{b'_1, \dots, b'_k\}$ with $c(b'_1) < \dots < c(b'_{r-k})$ is contracted and the elements in subset

$$E' := \{e \in E: c(e) \leq c(b'_{r-k})\} \setminus B'$$

are deleted. Then, a maximum-cost independent set $B'' := \{b''_1, \dots, b''_k\}$ with $c(b''_k) > c(b'_{r-k})$ and $c(b''_1) > \dots > c(b''_k)$, which is a base in matroid $M \setminus E'/B'$, is determined.

If, for a given matroid $M = (E, \mathcal{I})$, m denotes the cardinality of the ground set, the following result can be concluded from the previous observations.

Theorem 4 *Minus-plus Univ-MMBP is solvable by at most $\min\{m^k, m^{r-k}\}$ applications of Algorithm 1 to matroids of the type $M \setminus E'/B'$ or $M \setminus E''/B''$.*

Proof Follows from Lemma 3 and the subsequent remarks by going through all independent sets B' or B'' , respectively. Note that the minimum-cost independent sets found by Algorithm 1 must have cardinality $r - k$ or k in order to form a base $B^* \in \mathcal{B}$ together with the elements in subsets B' or B'' . \square

Lemma 3 and Theorem 4 can be carried over to *plus-minus Univ-MMBP* which is defined by swapping the non-negative and non-positive coefficients in the universal weight vectors. In plus-minus Univ-MMBP, the universal weight vectors are of the form

$$\lambda = (\alpha_1, \dots, \alpha_k, -\beta_{k+1}, \dots, -\beta_r).$$

Surprisingly, this adaptation from minus-plus to plus-minus Univ-MMBP is no longer correct for our results on uniform matroids (see Theorems 2 and 3). We will discuss this phenomenon as part of Univ-MMBP with multiple sign changes in the following section.

It should be noted that the result of Theorem 4 looks at first sight rather negative, since the numbers k and $r - k$ of the non-positive or non-negative weight coefficients occur in the exponents, thus making the running time of our algorithm exponential. But, on the other hand, these numbers are fixed for several specific instances of Univ-MMBP such that Theorem 4 yields a polynomial-time algorithm. In Table 1, examples for plus-minus Univ-MMBP

where k or $r-k$ are fixed, and, therefore, not part of the input, are the balanced, (k_1, k_2) -balanced and minimum deviation objective functions, respectively.

This solution approach can be sped up if the universal weight vector starts with a large block of zero weights, i.e.,

$$\lambda = (0, \dots, 0, -\alpha_{k_1+1}, \dots, -\alpha_{k_1+k_2}, \beta_{k_1+k_2+1}, \dots, \beta_r).$$

The idea is to add an independent set of cardinality k_1 , associated with the zero weight coefficients, to a base in $M \setminus E''/B''$. To formalize this, we define the matroid $M//T^k$ which will be introduced next using the concept of restriction matroids $M|T$ (see Appendix).

Definition 4 Given a matroid $M = (E, \mathcal{I})$ together with a subset $T \subseteq E$ and an integer $k \in \mathbb{N}$ such that there exists an independent set $I(M|T) \in \mathcal{I}(M|T)$ of cardinality k , the matroid $M//T^k$ has ground set $E(M//T^k) := E \setminus T$ and independence system

$$\mathcal{I}(M//T^k) := \{I \subseteq E(M//T^k) : M|T \text{ has an independent set } I(M|T) \text{ of cardinality } k \text{ such that } I \dot{\cup} I(M|T) \in \mathcal{I}\}.$$

Since $M//T^k$ is indeed a matroid, we can prove the following theorem. For details, we refer to Turner [23].

Theorem 5 *Let*

- a matroid $M = (E, \mathcal{I})$ and
- a weight vector $\lambda \in \mathbb{R}^r$ of minus-plus form starting with a block of zero weight coefficients be given.

Then, minus-plus Univ-MMBP is solvable by at most m^{k_2} applications of the min version of Algorithm 1 to matroids of the type $M \setminus E''/B''/E^{k_1}$.

Here, $B'' := \{b''_1, \dots, b''_{k_2}\}$ satisfying $c(b''_1) > \dots > c(b''_{k_2})$ are independent sets. We define

$$E'' := \{e \in E : c(b''_1) \geq c(e) \geq c(b''_{k_2})\} \setminus B''$$

and

$$E^{k_1} := \{e \in E : c(e) > c(b''_1)\},$$

respectively. The same works for minus-plus Univ-MMBP with a large block of zero weights at the end.

6 Universal Weights with Multiple Sign Changes

In this section, we study universal matroid base problems where the weight vectors have more than one sign change. We restrict ourselves to the case of uniform matroids and consider universal weight vectors with non-negative (non-positive) coefficients in the first (last) block. Without loss of generality, we assume that the costs $c(e)$, $e \in E$, are pairwise different.

Definition 5 A universal weight vector has *p-fixed sign changes form* if

$$\lambda = (\alpha_1, \dots, \underbrace{\alpha_{k_1-1}, -\beta_{k_1}, \dots, \alpha_{k_p-1}, -\beta_{k_p}}_{(+, -)}, \dots, -\beta_r)$$

has p fixed $(+, -)$ -sign changes at positions k_1, \dots, k_p where $\alpha_i, \beta_j \in \mathbb{R}_0^+$. The resulting universal matroid base problem is denoted as *p-fixed Univ-MMBP*.

For uniform matroids $M_{r,m} = (E(M_{r,m}), \mathcal{I}(M_{r,m}))$, p -fixed Univ-MMBP can be reduced to solving a sequence of universal matroid base problems in partition matroids. Given an independent set $B^p := \{b_{k_1}^p, \dots, b_{k_p}^p\}$ of cardinality p with $c(b_{k_1}^p) > \dots > c(b_{k_p}^p)$, the matroid $M_{r,m}^p$ has ground set $E(M_{r,m}^p) := E(M_{r,m})$ and independence system

$$\begin{aligned} \mathcal{I}(M_{r,m}^p) := \{ & I \subseteq E(M_{r,m}^p) : |I \cap B^p| \leq p \text{ and} \\ & |I \cap E(M_{r,m}^p)_i| \leq d_i \text{ for all } i = 1, \dots, p+1 \} \end{aligned}$$

where

$$\begin{aligned} E(M_{r,m}^p)_1 &:= \{e \in E(M_{r,m}^p) : c(e) > c(b_{k_1}^p)\}, \quad d_1 := k_1 - 1, \\ E(M_{r,m}^p)_{p+1} &:= \{e \in E(M_{r,m}^p) : c(e) < c(b_{k_p}^p)\}, \quad d_{p+1} := r - k_p, \end{aligned}$$

and

$$E(M_{r,m}^p)_i := \{e \in E(M_{r,m}^p) : c(b_{k_{i-1}}^p) > c(e) > c(b_{k_i}^p)\}, \quad d_i := k_i - k_{i-1} - 1,$$

for all $i = 2, \dots, p$.

By definition, a base $B \in \mathcal{B}(M_{r,m}^p)$ contains all p elements of subset B^p and d_i elements of subsets $E(M_{r,m}^p)_i$, $i = 1, \dots, p+1$, if $|E(M_{r,m}^p)_i| \geq d_i$. Since

$$p + d_1 + \sum_{i=2}^p d_i + d_{p+1} = p + (k_1 - 1) + \sum_{i=2}^p (k_i - k_{i-1} - 1) + (r - k_p) = r,$$

it is a base of the uniform matroid. Conversely, a base $B \in \mathcal{B}(M_{r,m})$ having the element b_i^p , $i = k_1, \dots, k_p$, as i th largest cost element is a base of the partition matroid. The independent set B^p together with the matroid $M_{r,m}^p$ can be discarded whenever $|E(M_{r,m}^p)_i| < d_i$ for some $i \in \{1, \dots, p+1\}$. A universal minimum-cost base $B^* \in \mathcal{B}(M_{r,m}^p)$ can be determined by taking all elements in subset B^p and solving $p+1$ universal matroid base problems in subsets $E(M_{r,m}^p)_i$ for all $i = 1, \dots, p+1$. Depending on the sign of the weight coefficients, this can be done by applying the min or max version of the greedy algorithm, or the combination of both as proposed in Theorem 2.

Thus, we obtain Theorem 6.

Theorem 6 *Let*

- a uniform matroid $M_{r,m} = (E(M_{r,m}), \mathcal{I}(M_{r,m}))$ and
- a weight vector $\lambda \in \mathbb{R}^r$ of *p-fixed sign changes form* be given.

Then, p -fixed Univ-MMBP is solvable by at most m^p Univ-MMBPs in partition matroids of the type $M_{r,m}^p = (E(M_{r,m}^p), \mathcal{I}(M_{r,m}^p))$.

Proof Follows from the preceding remarks by going through all independent sets $B^p := \{b_{k_1}^p, \dots, b_{k_p}^p\}$ associated with the negative weights $-\beta_{k_1}, \dots, -\beta_{k_p}$. \square

7 Applications to Universal Bipartite Matching and Shortest Path Problems

This section is concerned with two applications of transversal matroids to universal bipartite matching and shortest path problems. The goal is to identify special cases of (directed) graphs $G = (V, E)$ with suitable costs and topology such that the corresponding universal problem is solvable as Univ-MMBP.

Using the equivalence between (maximum cardinality) matchings and (maximal) partial transversals, we first show that special cases of universal bipartite matching problems can be solved by a greedy algorithm. For literature on bipartite matchings, we refer the reader to Lawler [14] or Schrijver [21].

Definition 6 Consider a bipartite graph $G = (V, E)$ with

- costs $c(e) \in \mathbb{R}$ for all $e \in E$ and
- weights $\lambda_i \in \mathbb{R}$ for all $i = 1, \dots, |M|$,

where $|M|$ is the size of the maximum cardinality matchings $M \subseteq E$ in graph G . Then, the *universal bipartite matching problem* (Univ-BMP) is

$$\max_{M \subseteq E} f_\lambda(M) := \sum_{i=1}^{|M|} \lambda_i \cdot c_{(i)}(M). \quad (4)$$

Observe that, in problem (4), the universal objective function is of the maximization type. Since the cardinality of the feasible solutions, i.e. the maximum cardinality matchings, is fixed, we can assume without loss of generality that $c(e) \geq 0$ for all edges $e \in E$.

Theorem 7 *Let*

- *weight coefficients $\lambda_1, \dots, \lambda_{|M|} \geq 0$ and*
- *a bipartite graph $G = (V, E)$ with vertices $V := V_1 \dot{\cup} V_2$, edges $E \subseteq V_1 \times V_2$ and costs $c(e) = c_i$, $c_i \in \mathbb{R}_0^+$, for all edges $e = [i, j] \in E$ incident to vertex $i \in V_1$ (see, e.g. Figure 3) be given.*

Then, Univ-BMP is solvable by the max version of Algorithm 1.

Proof In the transversal matroid $M(\mathcal{A}) = (E(M(\mathcal{A})), \mathcal{I}(M(\mathcal{A})))$ associated with graph G , we have that $S := V_1$ and $\mathcal{A} := \{A_j : j \in V_2\}$ where $A_j := \{s_i \in S : [i, j] \in E\}$. Recall that $E(M(\mathcal{A})) := S$ and

$$\mathcal{I}(M(\mathcal{A})) := \{I \subseteq E(M(\mathcal{A})) : I \text{ is a partial transversal of collection } \mathcal{A}\}.$$

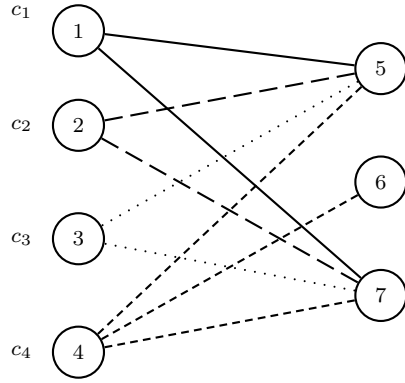


Fig. 3 Costs $c(e) = c_i$ for edges $e = [i, j]$ incident to vertex i

If we set the cost of element $i \in E(M(\mathcal{A}))$ equal to c_i , the sorted cost vector $c_{\geq}(B^*)$ of a maximum-cost base $B^* := \{b_1^*, \dots, b_r^*\}$, which is a maximal partial transversal of collection \mathcal{A} , equals the sorted cost vector $c_{\geq}(M^*)$ of the associated maximum cardinality matching $M^* := \{[b_1^*, j_1^*], \dots, [b_r^*, j_r^*]\}$. Note that this holds independently of which vertices j_1^*, \dots, j_r^* are matched to b_1^*, \dots, b_r^* and this observation is important since the subsets $A_{j_k^*}$ and, thus, the vertices j_k^* can, in general, not be chosen at the same time the elements b_k^* are chosen in the greedy algorithm (compare Oxley [18]). As a consequence, we have that $f_{\lambda}(B^*) = f_{\lambda}(M^*)$. The final claim follows since, conversely, a maximum cardinality matching induces a maximal partial transversal, i.e. a base in matroid $M(\mathcal{A})$, of the same universal cost. \square

Note that the solution of Univ-BMP is trivial when all vertices $i \in V_1$ can be matched. Then, any maximum cardinality matching is optimal. Furthermore, we can observe:

Corollary 2 *If $\lambda_1, \dots, \lambda_{|M|} \geq 0$ and $c_i > 0$ for all edges $e = [i, j] \in E$ incident to vertex $i \in V_1$, Univ-BMP can be solved by any bipartite weighted matching algorithm.*

Proof In this case, any maximum-weight matching has maximum cardinality. The induced maximal partial transversal is optimal for the sum problem and can be found by the max version of Algorithm 1 when breaking ties in favor of its elements. The optimality for Univ-BMP follows by Theorem 7. \square

Note that, for non-negative costs, a maximum-weight matching $M_k \subseteq E$ of cardinality k can always be extended to a maximum-weight matching $M^* \subseteq E$ of maximum cardinality by using augmenting paths. Concerning universal bipartite matching problems, we conclude with the following result:

Corollary 3 *If $G = (V, E)$ is a complete bipartite graph, p -fixed Univ-BMP having p fixed $(-, +)$ -sign changes at positions k_1, \dots, k_p can be solved by the procedure described in Theorem 6.*

Proof Follows since transversal matroids associated with complete bipartite graphs are uniform. \square

The second application is to universal shortest path problems with fixed length (for more general universal shortest path problems, see Turner [24, 25]).

Definition 7 Consider a digraph $G = (V, E)$ with source s and sink t having

- costs $c(e) \in \mathbb{R}$ for all $e \in E$ and
- weights $\lambda_i \in \mathbb{R}$ for all $i = 1, \dots, l$,

where we assume that all (s, t) -paths in digraph G have fixed length l . If the set of (s, t) -paths is denoted by \mathcal{P}_{st} , the *universal shortest path problem (Univ-SPP)* is

$$\min_{P \in \mathcal{P}_{st}} f_\lambda(P) := \sum_{i=1}^l \lambda_i \cdot c_{(i)}(P).$$

In this paper, we study the case of universal shortest paths in so-called lattice graphs, introduced below. For this purpose, we consider two directed lattice paths

$$P' := (p'_1, \dots, p'_{m+r}) \quad \text{and} \quad P'' := (p''_1, \dots, p''_{m+r})$$

with start point $(0, 0)$ and end point (m, r) which are defined on a lattice L . They have $m + r$ steps consisting of east steps $E = (1, 0)$ and north steps $N = (0, 1)$ such that the “lower” path P' does not exceed the “upper” path P'' . In total, the paths have m east steps and r north steps (see Figure 4).

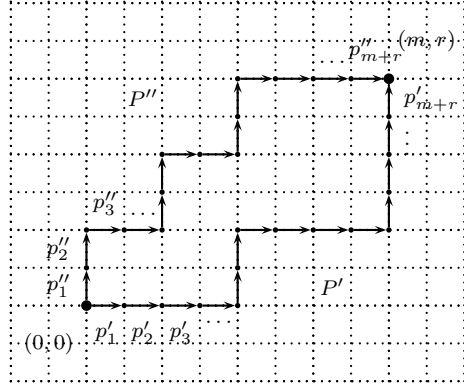


Fig. 4 Lattice paths P' and P''

With respect to paths P' and P'' , the lattice path matroid is introduced by Bonin and de Mier [1] or Bonin et al. [2]:

Definition 8 Given two lattice paths P' and P'' with north steps $p'_{u'_1}, \dots, p'_{u'_r}$ and $p''_{l'_1}, \dots, p''_{l'_r}$ where $u'_1 < \dots < u'_r$ and $l'_1 < \dots < l'_r$, respectively, such that path P' does not exceed path P'' . The *lattice path matroid* $M(P', P'')$ is the transversal matroid associated with the finite set $S := \{1, \dots, m+r\}$ and the collection $\mathcal{N} := \{N_1, \dots, N_r\}$ where the interval $N_i = [l'_i, u'_i]$ of integers is non-empty for all $i = 1, \dots, r$.

If the elements of the ground set are interpreted as steps $1, \dots, m+r$ of lattice paths P starting in point $(0,0)$ and ending in point (m,r) , any subset $X \subseteq E(M(P', P''))$, where $E(M(P', P'')) := S$, induces a path $P(X) := (p_1(X), \dots, p_{m+r}(X))$ with

$$p_j(X) := \begin{cases} N & \text{if } j \in X \\ E & \text{if } j \notin X. \end{cases}$$

The elements $j \in X$ are exactly the north steps $p_j(X)$ of lattice path $P(X)$. Using this interpretation, the bases $B \in \mathcal{B}(M(P', P''))$ – which are maximal partial transversals – can be identified with the lattice paths $P(B)$ between P' and P'' . They have cardinality r and the i th north step of the corresponding lattice paths is in the interval $N_i = [l'_i, u'_i]$ (see Bonin and de Mier [1]).

A lattice graph $G = (V, E)$ is the directed subgraph of a lattice L bounded by two lattice paths P' and P'' . The vertex set V contains all lattice points (i_1, i_2) in the region between these paths and the edge set E consists of all east and north steps connecting these lattice points, i.e. $E := E_H \cup E_V$, where the set E_H contains all *horizontal edges* $((i_1, i_2), (i_1 + 1, i_2))$ and the set E_V contains all *vertical edges* $((i_1, i_2), (i_1, i_2 + 1))$. Setting $s = (0, 0)$ and $t = (m, r)$, there is a one-to-one correspondence between the (s, t) -paths in graph G and the bases of matroid $M(P', P'')$. The paths $P \in \mathcal{P}_{st}$ have length $m+r$.

In order to apply the greedy algorithm to Univ-SPP, we require that all edges $e \in E$ which can be identified with an element $j \in E(M(P', P''))$ have the same costs. The edges $e \in E$ occurring in step j , $j = 1, \dots, m+r$, of lattice paths $P(B)$ form a diagonal as illustrated in Figure 5 by using different line styles (solid, solid and curved right, solid and curved left, dashed, dashed and curved right, dashed and curved left, or dotted).

Since the definition of lattice path matroids is based on north steps, the edges associated with an element $j \in E(M(P', P''))$ should be the vertical edges which occur in step j . The property that these edges have equal costs is denoted as *diagonal cost structure* (see Figure 6).

The costs of the horizontal edges must be defined such that they do not affect the universal objective function value. This holds if the horizontal edges have constant costs, or, more generally, if the costs of the horizontal edges associated with all steps $j \in E_i$ for each $i = 1, \dots, m$ are equal. The interval $E_i = [l'_i, u'_i]$ contains the i th east step of the lattice paths between P' and P'' .

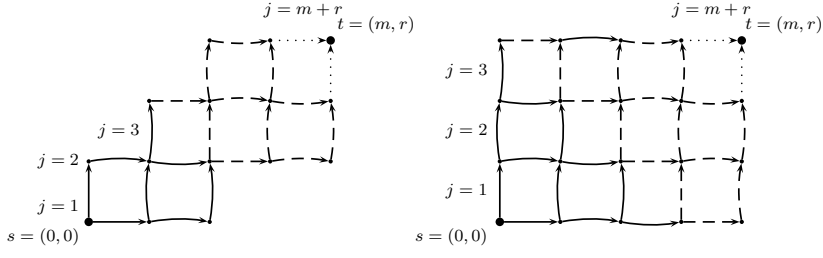


Fig. 5 Edges in steps $j = 1, \dots, m+r$

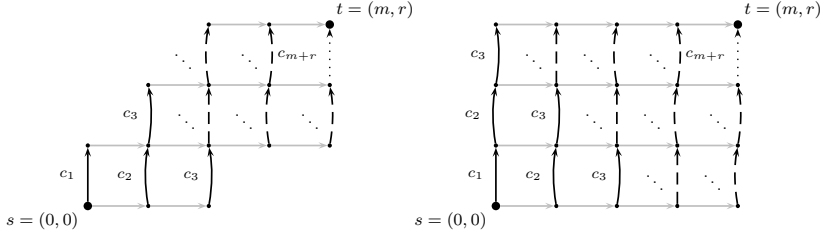


Fig. 6 Costs of vertical edges in steps $j = 1, \dots, m+r$

Theorem 8 *Let*

- *weight coefficients $\lambda_1, \dots, \lambda_{m+r} \geq 0$ and*
- *a lattice graph $G = (V, E)$ with diagonal cost structure for the vertical edges and costs $c(e) = c_i$, $c_i \in \mathbb{R}$, for the horizontal edges $e \in E$ associated with all steps $j \in E_i$ for each $i = 1, \dots, m$ (see, e.g. Figure 7) be given.*

Then, Univ-SPP is solvable by the min version of Algorithm 1.

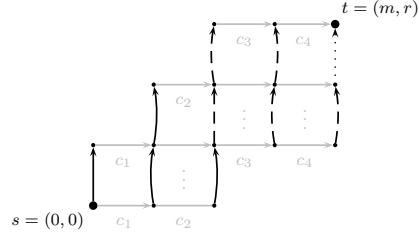


Fig. 7 Costs of horizontal edges associated with steps $j \in E_i$

Proof We prove this theorem for the special case of constant costs, that is, $c(e) = c$, $c \in \mathbb{R}$, for all horizontal edges $e \in E$.

By construction, any (s, t) -path P of lattice graph $G = (V, E)$ contains m horizontal edges which correspond to the east steps of the associated lattice path $P(B)$. Using the correspondence between the north steps of lattice path

$P(B)$ and the elements of base $B \in \mathcal{B}(M(P', P''))$, a minimum-cost base $B^* := \{b_1^*, \dots, b_r^*\}$ with $c(b_1^*) \geq \dots \geq c(b_r^*)$ defines a universal shortest path if we identify the elements b_1^*, \dots, b_r^* with some vertical edges in steps b_1^*, \dots, b_r^* . If $c < c(e)$ for all vertical edges $e \in E$, this is true since the sorted cost vectors of the paths $P \in \mathcal{P}_{st}$ have the form

$$c_{\geq}(P) = (c(b_r), \dots, c(b_1), \underbrace{c, \dots, c}_m)$$

where $B := \{b_1, \dots, b_r\}$ with $c(b_1) \leq \dots \leq c(b_r)$ is the associated base. By Lemma 2, it follows that

$$c_{(i)}(P(B^*)) \leq c_{(i)}(P(B))$$

for the lattice path $P(B^*)$ and any other lattice path $P(B)$. The argumentation is similar if $c > c(e)$ for all vertical edges $e \in E$ and in cases where $c \not\leq c(e)$ or $c \not\geq c(e)$. For (non-constant) costs $c(e) = c_i$, $c_i \in \mathbb{R}$, associated with all steps $j \in E_i$ for each $i = 1, \dots, m$, we use that any path $P \in \mathcal{P}_{st}$ has exactly one edge in each subset $E_i = [l'_i, u''_i]$. \square

In addition, we can state Corollary 4.

Corollary 4 *Univ-SPP can be solved by any shortest path algorithm.*

Proof Obviously, a lattice path $P(B^*)$ which corresponds to an optimal base $B^* \in \mathcal{B}(M(P', P''))$ according to Theorem 8 is a shortest path with respect to sum objective function. Conversely, we know that any minimum-cost base corresponding to a sum shortest path can be found by the greedy algorithm. This proves the claim. \square

If, for a fixed number of steps j_1, \dots, j_k , the associated horizontal edges have costs c_{j_1}, \dots, c_{j_k} (the vertical edges have diagonal cost structure and all remaining horizontal edges have constant costs), we use minors. A universal shortest path $P^* \in \mathcal{P}_{st}$ containing one of the horizontal edges associated with step j_l can be computed in the deletion matroid $M(P', P'') \setminus j_l$ provided that a base $B^*(M(P', P'') \setminus j_l) \in \mathcal{B}(M(P', P'') \setminus j_l)$ with r elements still exists. Conversely, a path containing none of these edges can be computed in the contraction matroid $M(P', P'')/j_l$. Thus, for the 2^k possibilities of including or excluding horizontal edges associated with steps j_1, \dots, j_k , we apply the greedy algorithm to minors

$$M(P', P'') \setminus \{j_{i_1}, \dots, j_{i_p}\} / \{j_{i_{p+1}}, \dots, j_{i_k}\}.$$

If we interchange the roles of east and north steps by setting

$$p_j^D(Y) := \begin{cases} E & \text{if } j \in Y \\ N & \text{if } j \notin Y, \end{cases}$$

we can define the *dual lattice path matroid* $M^D(P', P'')$ to be the transversal matroid associated with the finite set $S := \{1, \dots, m + r\}$ and the collection

$\mathcal{E} := \{E_1, \dots, E_m\}$. Since the bases of dual matroids are the complements of the bases of the given matroid, see Oxley [18], the elements of the bases $B^D \in \mathcal{B}(M^D(P', P''))$ correspond to the east steps of lattice paths $P^D(B^D)$ and to the horizontal edges in lattice graph G .

According to this, we obtain the dual lattice graph $G^D = (V^D, E^D)$ where the roles of horizontal and vertical edges are reversed. Theorem 8 remains valid if we have a diagonal cost structure for the horizontal edges and costs $c(e) = c_i$, $c_i \in \mathbb{R}$, for all vertical edges $e \in E$ associated with all steps $j \in N_i$ for each $i = 1, \dots, r$.

Together with directed sums (see Appendix), we can enlarge the class of digraphs for which universal shortest path problems can be solved.

Theorem 9 *Let*

- *weight coefficients $\lambda_1, \dots, \lambda_{m+r} \geq 0$ and*
- *a digraph $\tilde{G} = (\tilde{V}, \tilde{E})$ which is the concatenation of a lattice graph $G = (V, E)$ and a dual lattice graph $G^D = (V^D, E^D)$ having one intersection point (see, e.g. Figure 8) be given.*

If the vertical edges of the lattice graph $G = (V, E)$ and the horizontal edges of the dual lattice graph $G^D = (V^D, E^D)$ have diagonal cost structure and $c(e) = c_i$, $c_i \in \mathbb{R}$, for all horizontal edges $e \in E$ associated with steps $j \in E_i$ and all vertical edges $e \in E^D$ associated with steps $j \in N_i$, Univ-SPP is solvable by the min version of Algorithm 1.

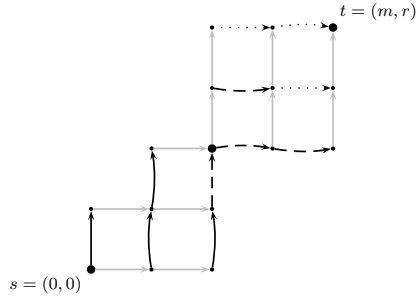


Fig. 8 Digraph $\tilde{G} = (\tilde{V}, \tilde{E})$

Proof Digraph $\tilde{G} = (\tilde{V}, \tilde{E})$ corresponds to the direct sum of a lattice path matroid $M_1(P'_1, P''_1)$ and a dual lattice path matroid $M_2^D(P'_2, P''_2)$. Since the bases of the direct sum $M_1(P'_1, P''_1) \oplus M_2^D(P'_2, P''_2)$ are the unions of the bases $B \in \mathcal{B}(M_1(P'_1, P''_1))$ and $B^D \in \mathcal{B}(M_2^D(P'_2, P''_2))$, the greedy algorithm can be applied to both matroids. Similar to Theorem 8, it follows that the resulting base (and the associated path) is optimal. \square

Square lattice path matroids result from paths

$$P' := (\underbrace{E, \dots, E}_m, \underbrace{N, \dots, N}_r) \quad \text{and} \quad P'' := (\underbrace{N, \dots, N}_r, \underbrace{E, \dots, E}_m).$$

For square lattice graphs which induce square lattice path matroids, we can show the following result:

Corollary 5 *If $G = (V, E)$ is a square lattice graph such that the costs of the horizontal edges associated with steps $j \in E_i$ are lower (upper) bounds on the costs of the vertical edges having diagonal cost structure, p -fixed Univ-SPP can be solved by the procedure described in Theorem 6.*

Proof Follows since square lattice path matroids are uniform. \square

Finally, we remark that the results of Theorem 7 (for Univ-BMP) and Theorem 8 (for Univ-SPP) carry over to non-positive weight coefficients by applying the alternative versions of Algorithm 1.

8 Conclusions and Further Remarks

In this paper, we introduced the concept of Univ-MMBP as a rather powerful model for matroid base problems with classical and new objective functions. Given this model, we illustrated the potential and limits of the well-known greedy algorithm and some extensions thereof to solve universal matroid base problems in uniform and non-uniform matroids. We proved the validity of these algorithms for problems with non-negative or non-positive weights as well as for problems with one (uniform and non-uniform matroids) or multiple (uniform matroids) sign changes in the universal weight vector. We obtained a characterization of uniform matroids by combining minimum-cost and maximum-cost bases found by the min and max version of the greedy algorithm. We studied applications to universal bipartite matching and shortest path problems in bipartite or lattice graphs with special cost structure.

Universal matroid base problems with weight vectors

$$\begin{aligned}
- \lambda &= (\underbrace{\alpha^1, \dots, \alpha^1}_{k-1}, -\beta_k, \alpha^2, \dots, \alpha^2) \\
- \lambda &= (\underbrace{\alpha^1, \dots, \alpha^1}_{k-1}, -\beta_{k+1}, \dots, -\beta_{k+q}, \alpha^2, \dots, \alpha^2) \\
- \lambda &= (\underbrace{\alpha^1, \dots, \alpha^1}_{k_1-1}, -\beta_{k_1}, \underbrace{\alpha^2, \dots, \alpha^2}_q, -\beta_{k_2}, \dots, -\beta_{k_{p-1}}, \underbrace{\alpha^p, \dots, \alpha^p}_{k_p-k_{p-1}-1}, -\beta_{k_p}, \alpha^{p+1}, \dots, \alpha^{p+1}) \\
- \lambda &= (\underbrace{\alpha^1, \dots, \alpha^1}_{k_1}, -\beta_{k_1+1}, \dots, -\beta_{k_1+q_1}, \underbrace{\alpha^2, \dots, \alpha^2}_{k_2-k_1-q_1}, -\beta_{k_2+1}, \dots, -\beta_{k_2+q_2}, \dots \\
&\quad \dots, -\beta_{k_{p-1}+1}, \dots, -\beta_{k_{p-1}+q_{p-1}}, \underbrace{\alpha^p, \dots, \alpha^p}_{k_p-k_{p-1}-q_{p-1}}, -\beta_{k_p+1}, \dots, -\beta_{k_p+q_p}, \alpha^{p+1}, \dots, \alpha^{p+1}),
\end{aligned}$$

where the number of negative weight coefficients $-\beta_i$ is fixed and the non-negative weight coefficients α^j in between occur in blocks of equal weights are investigated in Turner [23]. They can be interpreted as color-constrained optimization problems and are solved by adapting the matroid intersection algorithms proposed by Brezovec et al. [3,4] or Gabow and Tarjan [9].

Acknowledgements The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7-PEOPLE-2009-IRSES) under grant agreement n° 246647 (project OptALI). The research is supported by the New Zealand Government and the Federal Ministry of Education and Research, Germany, grant DSS Evac Logistic, FKZ 13N12229.

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Appendix

In this appendix, we summarize denotations as well as basic definitions and results from matroid theory used in the preceding paper. We mainly refer to the book of Oxley [18]. This part will be deleted in the published version of the paper.

Generalizing matrices and graphs, matroids can be characterized by a series of equivalent axioms derived from linear algebra and graph theory.

Note that we use $X + y := X \cup \{y\}$ and $X - z := X \setminus \{z\}$ for a subset X , an element $y \notin X$ or $z \in X$, respectively.

Definition 9 (Characterization by independent sets) A *matroid* is a pair $M = (E, \mathcal{I})$ with a finite set E - the *ground set* of cardinality m - and a collection $\mathcal{I} \subseteq 2^E$ of subsets - the *independence system* - such that the following conditions hold:

- (a) $\emptyset \in \mathcal{I}$.
- (b) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.
- (c) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, there exists an element $e \in I_2 \setminus I_1$ such that $I_1 + e \in \mathcal{I}$.

Definition 9 (b) and (c) are known as *independent set* or *extension property*. Subsets $I \in \mathcal{I}$ are called *independent sets*.

In contrast, the *dependent sets* of matroid $M = (E, \mathcal{I})$ are the subsets $X \subseteq E$ that are not contained in the independence system. A matroid can, equivalently, be defined by means of its minimal (with respect to set inclusion) dependent sets which are called *circuits*. Definition 10 (b) and (c) are denoted as *non-inclusion* or *circuit-union property*.

Definition 10 (Characterization by circuits) Let E be a finite set and $\mathcal{C} \subseteq 2^E$ be a collection of subsets. Then, \mathcal{C} is the collection of circuits of a matroid $M = (E, \mathcal{C})$ if and only if the following conditions hold:

- (a) $\emptyset \notin \mathcal{C}$.
- (b) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.
- (c) If $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$ and $e \in C_1 \cap C_2$, there exists a subset $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

Besides these characterizations, there exist alternative axioms to uniquely determine a matroid. Among these are the *bases*, the *rank* and the *closure* (see Welsh [26]). With respect to the universal matroid base problems considered in this paper, we now give a description in terms of matroid bases.

Definition 11 (Characterization by bases) Let E be a finite set and $\mathcal{B} \subseteq 2^E$ be a collection of subsets. Then, \mathcal{B} is the collection of bases of a matroid $M = (E, \mathcal{B})$ if and only if the following conditions hold:

- (a) $\mathcal{B} \neq \emptyset$.
- (b) If $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 \setminus B_2$, there exists an element $b_2 \in B_2 \setminus B_1$ such that $B_1 - b_1 + b_2 \in \mathcal{B}$.

The bases are the maximal independent sets with respect to set inclusion, i.e. all proper supersets of bases $B \in \mathcal{B}$ are dependent. In particular, we have that $B + e$, where $e \notin B$, contains a unique circuit $C(e, B)$ with $e \in C(e, B)$. This is the *fundamental circuit* with respect to base $B \in \mathcal{B}$. Definition 11 (b) is the *base-swapping property* and describes how to swap between different bases. Another crucial result is the equi-cardinality of the matroid bases meaning that $|B_1| = |B_2|$ for two bases $B_1, B_2 \in \mathcal{B}$. This cardinality is the rank of matroid $M = (E, \mathcal{I})$, and is usually denoted by r .

In Section 3, we have already learned that the greedy algorithm computes minimum-cost bases for all cost functions $c: E \rightarrow \mathbb{R}$ defined on the ground set, but even more intriguing is the fact that this greedy approach only works if the problem structure is that of a matroid. This provides the last characterization of matroids which we would like to mention in this section.

Definition 12 (Definition by the greedy algorithm) Let E be a finite set and $\mathcal{I} \subseteq 2^E$ be a collection of subsets. Then, $M = (E, \mathcal{I})$ is a matroid if and only if the following conditions hold:

- (a) $\emptyset \in \mathcal{I}$.
- (b) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.
- (c) For all cost functions $c: E \rightarrow \mathbb{R}$, the greedy algorithm finds a minimum-cost base $B^* \in \mathcal{B}$.

The matroids associated with Definitions 9 and 10 are the *vector matroid* of a matrix $A \in \text{Mat}(n \times m, \mathbb{F})$ over a field \mathbb{F} and the *cycle matroid* of an undirected graph $G = (V, E)$ with vertex set V of cardinality n and edge set E of cardinality m , respectively. Other classes of matroids, which are considered in this paper, are *uniform*, *partition* and *transversal matroids*.

Definition 13 (Uniform matroid) Given two integers $r, m \in \mathbb{N}$ with $r \leq m$, the matroid $M_{r,m}$ with ground set $E(M_{r,m})$ of cardinality m and independence system

$$\mathcal{I}(M_{r,m}) := \{I \subseteq E(M_{r,m}) : |I| \leq r\}$$

is called a *uniform matroid*.

Partition matroids are, for instance, defined in Lawler [14].

Definition 14 (Partition matroid) Given a finite set $E := E_1 \dot{\cup} \dots \dot{\cup} E_n$ partitioned into subsets E_1, \dots, E_n and non-negative integers $d_1, \dots, d_n \in \mathbb{N}_0$, the matroid M with ground set E and independence system

$$\mathcal{I} := \{I \subseteq E: |I \cap E_i| \leq d_i \text{ for all } i = 1, \dots, n\}$$

is called a *partition matroid*.

Transversal matroids are associated with the partial transversals of a finite set $S := \{s_1, \dots, s_m\}$ and a collection $\mathcal{A} := \{A_1, \dots, A_n\} \subseteq 2^S$ of subsets. By definition, a *partial transversal* is a subset $S_k := \{s_{i_{j_1}}, \dots, s_{i_{j_k}}\} \subseteq S$ such that $s_{i_{j_1}} \in A_{j_1}, \dots, s_{i_{j_k}} \in A_{j_k}$ for some subsets $A_{j_1}, \dots, A_{j_k} \in \mathcal{A}$. A *transversal* is a subset $S_n := \{s_{i_1}, \dots, s_{i_n}\} \subseteq S$ such that $s_{i_1} \in A_1, \dots, s_{i_n} \in A_n$. The elements of the (partial) transversals are assumed to be distinct. We define:

Definition 15 (Transversal matroid) Given a finite set $S := \{s_1, \dots, s_m\}$ and a collection $\mathcal{A} := \{A_1, \dots, A_n\} \subseteq 2^S$ of subsets, the matroid $M(\mathcal{A})$ with ground set $E(M(\mathcal{A})) := S$ and independence system

$$\mathcal{I}(M(\mathcal{A})) := \{I \subseteq E(M(\mathcal{A})): I \text{ is a partial transversal of collection } \mathcal{A}\}$$

is called a *transversal matroid*.

For a given matroid $M = (E, \mathcal{I})$, new matroids $M' = (E', \mathcal{I}')$ can be obtained by the techniques of *dualization*, *truncation*, *deletion* and *contraction*. Similar constructions lead to *direct sums*, *parallel* or *series connections*, *single-element extensions* and *quotients* of matroids. The most important operations when dealing with universal matroid base problems are deletions, also called restrictions, and contractions.

Definition 16 (Deletion, restriction and contraction) Given a matroid $M = (E, \mathcal{I})$ and a subset $T \subseteq E$, we define

- (a) the *deletion* of subset T as $M \setminus T$ with $E(M \setminus T) := E \setminus T$ and

$$\mathcal{I}(M \setminus T) := \{I \subseteq E(M \setminus T): I \in \mathcal{I}\}.$$

This is equal to the *restriction* to subset $E \setminus T$ and can, alternatively, be denoted as $M|(E \setminus T) = (E(M|(E \setminus T)), \mathcal{I}(M|(E \setminus T)))$.

- (b) the *contraction* of subset T as M/T with $E(M/T) := E \setminus T$ and

$$\begin{aligned} \mathcal{I}(M/T) := \{I \subseteq E(M/T): M|T \text{ has a base } B(M|T) \\ \text{such that } I \dot{\cup} B(M|T) \in \mathcal{I}\}. \end{aligned}$$

A matroid of the form $(M \setminus T_1)/T_2 = (M/T_2) \setminus T_1$ where $T_1 \cap T_2 = \emptyset$ obtained by a sequence of deletions, i.e. restrictions, and contractions is called a *minor* of matroid $M = (E, \mathcal{I})$. Note that the order of these operations is arbitrary, that is, $M \setminus T_1/T_2 = M/T_2 \setminus T_1$.

For a given matroid $M = (E, \mathcal{B})$, the complements of the bases $B \in \mathcal{B}$ define the collection of bases of the dual matroid $M^D = (E^D, \mathcal{B}^D)$.

Definition 17 (Dual matroid) Given a matroid $M = (E, \mathcal{B})$ with ground set E and collection of bases \mathcal{B} , the matroid M^D with ground set $E^D := E$ and collection of bases

$$\mathcal{B}^D := \{B^D \subseteq E^D : B^D = E \setminus B \text{ for some base } B \in \mathcal{B}\}$$

is called the *dual matroid* of matroid M .

The direct sum of two matroids is defined as follows:

Definition 18 (Direct sum) Given two matroids $M_1 = (E(M_1), \mathcal{I}(M_1))$ and $M_2 = (E(M_2), \mathcal{I}(M_2))$ with disjoint ground sets $E(M_1)$ and $E(M_2)$ as well as independence systems $\mathcal{I}(M_1)$ and $\mathcal{I}(M_2)$, the matroid $M_1 \oplus M_2$ with ground set

$$E(M_1 \oplus M_2) := E(M_1) \dot{\cup} E(M_2)$$

and independence system

$$\begin{aligned} \mathcal{I}(M_1 \oplus M_2) := & \{I \subseteq E(M_1 \oplus M_2) : \\ & I = I_1 \dot{\cup} I_2 \text{ for } I_1 \in \mathcal{I}(M_1) \text{ and } I_2 \in \mathcal{I}(M_2)\} \end{aligned}$$

is called the *direct sum* of matroids M_1 and M_2 .